Stochastic differential equation and some of its applications

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 Abstract:

Stochastic differential equation (SDE) plays important roles in the real life specially at stock price and physical systems subject. The purpose of this paper, is provide an introduction to the stochastic differential equation and some of its applications. Black-scholes partial differential equation is studied, which depended on SDE.

الملخص:

تلعب المعادلات التفاضلية الجزئية العشوائية دورا مهما في حياتنا العملية وخاصة في معرفة أسعار الأسهم التجارية، والأنظمة الفيزيائية.   إن الهدف من هذه الورقة البحثية هو عرض و تقديم المفاهيم الأساسية للمعادلات الجزئية العشوائية وتوضيح بعض من تطبيقاتها المهمة. حيث قمنا بدراسة المعادلة التفاضلية الجزئية لبلاك سكولس والتي تعتمد علي المعادلات التفاضلية الجزئية العشوائية.

1. Introduction

The stochastic process is a natural model for describing the evolution of real-life processes, objects and systems in time and space. A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process..Typically, SDEs contain a variable which represents random white noise calculated as the derivative of Brownian motion.

Definition (1) [6]

A stochastic differential equation is a differential equation whose coefficients are random numbers or random functions of the independent variable (or variables). Just as in normal differential equations, the coefficients are supposed to be given, independently of the solution that has to be found. Hence stochastic differential equations are the appropriate tool for describing systems with external noise

To explain the stochastic differential equation let us begin with an ordinary differential equation

$$\frac{dx(t)}{dt} = f(t,x), \quad dx(t) = f(t,x)dt, \qquad (1)$$

With initial condition $x(0) = x_0$ can be written in integral form as

$$x(t) = x_0 + \int_0^t f(s, x(s))ds, \qquad (2)$$

Where $x(t) = x(t, x_0, t_0)$ the solution with initial condition $x(t_0) = x_0$
We can now substitute the right hand side of the equation (1) for x inside the integral, which gives

$$x(t) = x_0 + \int_0^t f\left(x_0 + \int_0^t f(s, x(s))ds,\right)ds, \qquad (3)$$

$$x(t) = x_0 + fx_0 \int_0^t ds + \int_0^t\int_0^t f^2(s + x(s))ds^2, \qquad (4)$$

$$x(t) = x_0 + fx_0 t + \int_0^t\int_0^t f^2(s + x(s))ds^2 \qquad (5)$$

Doing the same substitution inside the last integral Eq. (5) get

$$x(t) = x_0 + fx_0 t + \int_0^t\int f^2\left(s + \int_0^t f(s, x(s)ds\right)ds^2 \ (6)$$

$$x(t) = x_0 + fx_0 t + fx_0 \frac{t^2}{2} + \int\int\int_0 f^3(s + x(s))ds^3 \qquad (7)$$

Doing the same another substituted we get

$$x(t) = x_0 + fx_0t + fx_0\frac{t^2}{2} + fx_0\frac{t^3}{6} + \ldots \qquad (8)$$

$$x(t) = \left(1 + ft + \frac{f^2t^2}{2!} + \frac{f^3t^3}{3!} + \ldots\right)x_0 \qquad (9)$$

We can see that the series in Eq. (9) its Taylor series for $\exp(ftx_0)$, as the series is converges so we arrived to the solution

$$x(t) = \exp(ft)x_0$$

The homogonous linear Eq.(1) can be generalized to the multidimensional equation as following

$$\frac{dX}{dt} = Fx, X_0 = given \qquad (10)$$

Where F is a constant (time-independent) matrix. The solution of this equation gives as below

$$X(t) = \left(I + Ft + \frac{F^2t^2}{2!} + \frac{F^3t^3}{3!} + \ldots\right)X_0 \qquad (11)$$

Which can be written as

$$X(t) = \exp(Ft)dX_0 \qquad (12)$$

2. Solution of the general linear stochastic differential equation

We will explain solution of the matrix exponential equation which is time varying homogenous equation that is given in the following equation

$$\frac{dX}{dt} = F(t)X, X_0 = given \qquad (13)$$

Assume the solution giving as

$$X(t) = \psi(t,t_0)X(t_0) \qquad (14)$$

Where $\psi(t,t_0)$ is the transition matrix which is defined via the properties

$$\frac{\partial \psi(\tau,t)}{\partial \tau} = F(\tau)\psi(\tau,t)$$

$$\frac{\partial \psi(\tau,t)}{\partial t} = -\psi(\tau,t)F(t)$$

$$\psi(\tau,t) = \psi(\tau,s)\psi(s,t)$$

$$\psi(t,\tau) = \psi^{-1}(\tau,t)$$

$$\psi(t,t) = I$$

(15)

Given the transition matrix we can then construct the solution to the inhomogeneous equation

$$\frac{dX(t)}{dt} = F(t)X(t) + L(t)w(t), X_0 = given \ (16)$$

$L(t)$ is a matrix $w(t)$ is a forcing function
its solution is

$$X(t). = \psi(t,t_0)X(t_{0)} + \int_{t_0}^{t}\psi(t,\tau)L(\tau)w(\tau)d\tau \ (17)$$

Where the integrating factor is $\psi(t_0,t)$

3. Stochastic differential equation and stock price

Stochastic differential equation play big role on determining the stock price, in this paper we will study Black-scholes option pricing as an example of using SDE in stock price

Definition 2 [6] A Brownian motion (also called Wiener process) is a continuous time stochastic process $B = \{B_t, t \geq 0\}$ characterized by the following three probabilities

1. Normal Increment: if $0 \leq s < t < \infty$, $B(t) - B(s)$ has a normal distribution with mean zero and variance $t - s$. if $s = 0$, then $B(t) - B(0)$ has normal distribution with zero mean and variance t.

2. Independence of Increments: $B(t) - B(s)$ is independent of the path.

3. Continuity of Paths ' no jumps ': $B(t)$, $t > 0$ are continuous functions of t.

3.1   Ito's Lemma: [1] [8]

Ito's lemma (also named Ito's formula) is a fundamental concepts of the stochastic differential equations. There are different important versions of the Ito's lemma, here we only present the Ito's lemma with space and time variables. For other versions see [1].

Let $S(t)$ be a function in t satisfies the follows stochastic differential equation

$$dS(t) = \mu(t)Sdt + \sigma(t)SdW, \qquad (1)$$

where $\mu(t)$ and $\sigma(t)$ are two deterministic functions, and W is a standard Brownian motion.

$$\left(dS(t)\right)^2 = \left(\mu(t)Sdt + \sigma(t)SdW\right)^2,$$
$$= \mu^2 S^2 \left(dt\right)^2 + \sigma^2 S^2 \left(dW\right)^2 + 2\mu\sigma S^2 dtdW.$$

From rules of the stochastic differential equations we have

$$\left(dt\right)^2 = 0, \ \left(dtdW\right) = 0, \ \text{and} \ \left(dW\right)^2 = dt.$$

Thus

$$dS^2(t) = \sigma^2 S^2 dt. \qquad (2)$$

Let $f(t,S)$ be a function twice continuously differentiable of S and continuously differentiable of t. Use Taylor expansion yield

$$df(t,S) = f_t dt + f_S dS + \frac{1}{2} f_{tt} \left(dt\right)^2 + \frac{1}{2} f_{SS} \left(dS\right)^2 + f_{tS} dtdS. \qquad (3)$$

Plugging Eq.(1) and Eq.(2) into Eq. (3) we get

$$df(t,S) = f_t dt + f_S \left(\mu(t)Sdt + \sigma(t)SdW\right) + \frac{1}{2} f_{SS} \left(\sigma^2 S^2 dt\right).$$

That can be simplified to

$$df(t,S) = \left(f_t + \mu Sf_S + \frac{1}{2}\sigma^2 S^2 f_{SS}\right)dt + \sigma Sf_S dW. \qquad (4)$$

The above equation is the Ito's formula, we will show later the use of this formula to derive the Black-Scholes partial differential equation for European options.

One important example of Ito's formula is option pricing function.

4 Black-Scholes partial differential equation

Black-Scholes model is a mathematical description of the financial market and derivatives, and the solution of this model (PDE) describes the value of this derivatives.

The model suggested by Black and Scholes to describe the behavior of prices is a continuous-time model with a risk-free asset also called a bond (with price B_t at time t), and one risky asset also called a stock (with price S_t at time t).

In the Black-Scholes model, the price of the risk-free asset B_t, satisfies the following ordinary differential equation

$$dB_t = rB_t dt, \qquad (5)$$

solution of this equation is

$$B_t = B_0 e^{rt}. \ (6)$$

This is mean, an investment of B_0 in a bond yields an amount $B_0 e^{rt}$ at time t. The risk-free asset always have a positive return.

The stock price S_t of the underlying asset of the option satisfies the following stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad (7)$$

where W_t is a standard Brownian motion, μ is known as a drift which is a measure of the average rate of growth of the asset price, and $\sigma > 0$ is a volatility of the stock price which is a measure of the riskiness of the asset, so the larger σ, the larger the fluctuations of S_t. Both of μ and σ are assumed

to be constants. The model is valid on the interval $[0,T]$, where T is the maturity of the option. If we assume $\sigma = 0$, then the stochastic differential equation becomes normal differential equation has the well-known solution $S_t = S_0 e^{\mu t}$, this shows that, when the volatility is equal to zero, the asset grows at a continuously compounded rate of the drift per a unit of time. So only when $\sigma > 0$, we obtain a randomly perturbed exponential function. The stock can have a positive or negative return.

Using Ito's Lemma solution of Eq. (7) is the geometric Brownian motion

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t},$$

where S_0 is the value of the underlying asset at time zero. S_t has a unique solution, and it's always positive since $S_0 > 0$. Eq.(7) is one of few stochastic differential equations that have an explicit solution

assume $U(t,s)$ is a function describing the stock price at time t, Using Ito's Lemma Eq.(4) we get

$$dU(t,S) = \left(\frac{\partial U}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 U}{\partial S^2} + \mu S\frac{\partial U}{\partial S}\right)dt + \sigma S\frac{\partial U}{\partial S}\,dW. \qquad (8)$$

Construct a portfolio Π of one option U and a number $-\Delta$ of the underlying assets, value of this portfolio is:

$$\Pi = U(t,S) - \Delta S, \qquad (9)$$

and the change in the value of this portfolio over a small time interval dt is

$$d\Pi = dU(t,S) - \Delta dS, \qquad (10)$$

$$d\Pi = \left(\frac{\partial U}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 U}{\partial S^2} + \mu S\frac{\partial U}{\partial S} - \mu S\Delta\right)dt + \sigma S\left(\frac{\partial U}{\partial S} - \Delta\right)dW, (11)$$

to eliminate the random term (dW) from the above equation choose $\Delta = \dfrac{\partial U}{\partial S}$, then Eq.(11) simplified to

$$dΠ = \left(\frac{\partial U}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 U}{\partial S^2} \right) dt. \qquad (12)$$

$\Delta = \dfrac{\partial U}{\partial S}$ is called Δ-hedging.

The portfolio now is completely riskless asset since it does not contain the stochastic term, so its return should be equal to the return of the riskless asset

$$dΠ = rΠdt,$$

$$\left(\frac{\partial U}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 U}{\partial S^2} \right) dt = r \left(U - S \frac{\partial U}{\partial S} \right) dt. (13)$$

The above equation is simplified to

$$\frac{\partial U}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 U}{\partial S^2} + rS \frac{\partial U}{\partial S} - rU = 0. (14)$$

This is the Black-Scholes PDE for European option with a boundary condition depends on the stock price at expiration date given by

$$U(T, S) = \varphi(S),$$

where $\varphi(S)$ is the payoff function depends on the type of the European options.

5   Solution of the Black-Scholes PDE

To solve Black-Scholes PDE of the European options we shall propose the following transformations, which allow us to transform the PDE (14) to a simpler parabolic equation. Firstly, consider the following transformation [4].

$$U(t, S) = e^{rt} f(t, x), \ x = \ln \frac{S}{k},$$

Eq. (14) is transformed to

$$\frac{\partial f}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\sigma^2}{2} - r\right)\frac{\partial f}{\partial x} = 0. \quad (15)$$

For further simplify makes another changes

$$f(t,x) = g(\tau, z),\ z = x - \left(\frac{\sigma^2}{2} - r\right)\tau,\ \tau = T - t,$$

under the above changes Eq. (15) becomes

$$\frac{\partial g}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 g}{\partial z^2},\qquad (16)$$
$$g(0,z) = g_0(z) = k\left(e^x - 1\right).$$

The above equation is structured. Therefore, the starting time $(\tau = 0)$ is at expiration date, and the equation will be solved backwards in time until $t = T$.

Solution of Eq. (16) can be written as

$$g(\tau, z) = \frac{1}{\sigma\sqrt{2\pi\tau}} \int\limits_{-\infty}^{\infty} g_0(y)\exp\left(\frac{(z-y)^2}{2\sigma^2\tau}\right)dy, \qquad (17)$$

thus, the final solution of the European call option PDE is given by

$$U(t,S) = SN(d_1) - ke^{-r(T-t)}N(d_2), (18)$$

where

$$d_1 = \frac{\ln\left(\dfrac{S}{k}\right) + \left(r + \dfrac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln\left(\dfrac{S}{k}\right) + \left(r - \dfrac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

and *N(d)* is a standard normal cumulative distribution function defined by

$$N(d) = \int_{-\infty}^{d} \exp\left(-\frac{1}{2}y^2\right) dy.$$

6   Black-Scholes PDE with Time-Dependent Parameters

For more generalization and realistic condition of the Black-Scholes model suppose all of the parameters in the Black-Scholes PDE Eq.(14) are time-varying parameters instead of constants, since in the real world all of these parameters change over time. [4]

$$\frac{\partial U}{\partial t} + \frac{\sigma^2(t)}{2}S^2\frac{\partial^2 U}{\partial S^2} + \left[r(t) - d(t)\right]S\frac{\partial U}{\partial S} - r(t)U = 0, \qquad (19)$$

to solve the above equation apply the following transformation

$$g(\tau, x) = \exp\left(\int_0^\tau r(\tau')d\tau'\right)U(t,S), \ x = \ln S,$$

this leads to the following equation

$$\frac{\partial g}{\partial \tau} = \frac{\sigma^2(\tau)}{2}x^2\frac{\partial^2 g}{\partial x^2} + \left[r(\tau) - d(\tau) - \frac{\sigma^2(\tau)}{2}\right]\frac{\partial g}{\partial x}. \qquad (20)$$

The fundamental solution of the above equation is

$$F(\tau, x) = \frac{1}{\sqrt{2\pi\int_0^\tau \sigma^2(u)du}}\exp\left(\frac{-\left\{x + \int_0^\tau\left[r(u) - q(u) - \dfrac{\sigma^2(u)}{2}\right]du\right\}^2}{2\int_0^\tau \sigma^2(u)du}\right), \quad (21)$$

solution of Eq. (21) can be written as

$$g(\tau, x) = \int_{-\infty}^{\infty} g(0, y) F(\tau, x - y) dy. \quad (22)$$

Thus the final solution of the Black-Scholes PDE with time-dependent parameters is

$$U(t, S) = SN\left(\tilde{d}_1\right) \exp\left(\int_{t}^{T} q(u) du\right) - kN\left(\tilde{d}_2\right) \exp\left(\int_{t}^{T} r(u) du\right), \quad (23)$$

where

$$\tilde{d}_1 = \frac{1}{\sqrt{\int_{t}^{T} \sigma^2(u) du}} \left(\ln \frac{S}{k} + \int_{t}^{T} \left[r(u) - q(u) - \frac{\sigma^2(u)}{2} \right] du \right),$$

$$\tilde{d}_2 = \tilde{d}_1 - \sqrt{\int_{t}^{T} \sigma^2(u) du}.$$

Conclusion

Stochastic differential equations SDE are very important like partial differential equations, however, we don't see many new researches in SDE compared to PDE, so in this work we study SDE in general, and show the importance of the SDE in deferent applications.

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